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Integral representations for the eigenfunctions of quantum open and periodic Toda chains from the QISM formalism

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Abstract

The integral representations for the eigenfunctions of N particle quantum open and periodic Toda chains are constructed within the framework of the quantum inverse scattering method. Both periodic and open N-particle solutions have essentially the same structure, being written as a generalized Fourier transform over the eigenfunctions of the $N-1$ particle open Toda chain with the kernels satisfying the Baxter equations of second and first order, respectively. In the latter case this leads to recurrent relations which result in a representation of Mellin–Burnes-type solutions of an open chain. As a byproduct, we obtain the Gindikin–Karpelevich formula for the Harish–Chandra function in the case of the $GL(N, \mathbb{R})$ group.

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1. Introduction

This paper is devoted to the well known quantum mechanical problem of finding the simultaneous eigenfunction of a commuting set of Hamiltonians for the periodic Toda chain. The first important step in this direction was made by Gutzwiller [1] who solved the problem for the particular cases of $N = 2$, 3 and 4 particles and found such important phenomena as quantization of the spectrum and separation of the multidimensional Baxter equation into the product of one-dimensional equations. In fact, he performed the quantization of the periodic Toda chain in terms of separated variables introduced by Flaschka and McLaughlin [2]. The next important step was taken by Sklyanin [3] who constructed an R-matrix formalism for both classical and quantum cases of Toda chains and introduced the algebraic method of separation variables for an arbitrary number of particles. His approach drastically simplifies the derivation of the Baxter equation and works for a wide spectrum of integrable models [4].

Our method of solving the spectral problem consists of an analytical re-interpretation of Sklyanin's algebraic ideas which allows one to find the integral representation for the eigenfunctions of the periodic Toda chain as a kind of generalized Fourier transform with eigenfunctions for the open Toda chain [5]. In turn, this method can be treated as a natural generalization of Gutzwiller's original approach. The explicit solution for the eigenfunctions of the open Toda chain plays a key role in this construction.

It has been discovered by Kostant [6] that the commuting set of Hamiltonians of an open Toda chain coincides with the Whittaker model of the centre of a universal enveloping algebra. Hence, the Whittaker functions are, in fact, eigenfunctions for the open Toda chain. In the usual group-theoretical way the Whittaker function is defined as a matrix element between compact and Whittaker vectors [7–9] in the principal series representation. There are many obstacles to generalizing this approach to other quantum models or to loop groups.

The present approach to constructing the eigenfunctions for both periodic and open chains is rather different [5, 10]: it is based on the quantum inverse scattering method for the *periodic* Toda chain [3]. One of the interesting results of analytical calculations in the R-matrix framework is the revealing of a recurrent relation between N and $N-1$ particle eigenfunctions for the *open* Toda chain (in fact, the idea of using a recurrent relation was pointed out by Sklyanin in [11]; our recurrent relations are an explicit realization of such an idea). This naturally leads to a new integral representation for the Weyl invariant Whittaker functions compared with classical results [7–9]. This representation is quite explicit and very useful for investigating the different asymptotics. In particular, the Gindikin–Karpelevich formula [12] for the Harish–Chandra function [13] can be obtained in a very simple way for the particular case of the $GL(N,\mathbb{R})$ group. The eigenfunctions for the periodic Toda chain are constructed in a rather explicit form and have essentially the same form as the recurrent relation mentioned above. The integral formula for eigenfunctions can be considered as a representation of the Whittaker functions for the $\widehat{GL}(N)$ group at the critical level.

The present approach can be generalized to other quantum integrable models. For example, a relativistic Toda chain is considered in [14].

2. The quantum Toda chain: a description of the model

2.1. Periodic spectral problem

The quantum N-periodic Toda chain is a multi-dimensional eigenvalue problem with N mutually commuting Hamiltonians $H_k(x_1, p_1; \ldots; x_N, p_N)$, $(k = 1, \ldots, N)$, where the simplest Hamiltonians have the form

$$\begin{aligned} H_1 &= \sum_{k=1}^{N} p_k \\ H_2 &= \sum_{k<m} p_k p_m - \sum_{k=1}^{N} \mathrm{e}^{x_k - x_{k+1}} \\ H_3 &= \sum_{k<m<n} p_k p_m p_n + \cdots \end{aligned} \tag{2.1}$$

$(x_{N+1} \equiv x_1)$, etc and the phase variables x_k, p_k satisfy the standard commutation relations $[x_k, p_m] = \mathrm{i}\hbar\delta_{km}$. The main goal is to find the solution to the eigenvalue problem

$$H_k \Psi_{\boldsymbol{E}} = E_k \Psi_{\boldsymbol{E}} \qquad k = 1, \ldots, N \tag{2.2}$$

with fast decreasing wavefunction $\Psi_{\boldsymbol{E}}$. To be more precise, let us note that, due to translation invariance, the solution to (2.2) has the following structure:

$$\Psi_{\boldsymbol{E}}(x_1, \ldots, x_N) = \widetilde{\Psi}_{\boldsymbol{E}}(x_1 - x_2, \ldots, x_{N-1} - x_N) \exp\left\{ \frac{\mathrm{i}}{\hbar} E_1 \sum_{k=1}^{N} x_k \right\}. \tag{2.3}$$

One needs to find the solution to (4.11) such that $\widetilde{\Psi}_{\boldsymbol{E}} \in L^2(\mathbb{R}^{N-1})$. In equivalent terms, we impose the requirement

$$\int f(E_1)\Psi_{\boldsymbol{E}}(x_1,\ldots,x_N)\,\mathrm{d}E_1 \in L^2(\mathbb{R}^N) \tag{2.4}$$

for any smooth function $f(y)$, $(y \in \mathbb{R})$ with finite support.

2.2. $GL(N-1,\mathbb{R})$ spectral problem

It turns out that the solution to (2.2) and (2.4) can be effectively written in terms of the wavefunctions corresponding to an *open* $(N-1)$-particle Toda chain (quantum $GL(N-1,\mathbb{R})$ chain). The Hamiltonians of the latter systems can be derived from (2.1) by letting formally $p_N = 0$, $x_N = \infty$, thus obtaining exactly $N-1$ commuting Hamiltonians $h_k(x_1,p_1;\ldots;x_{N-1},p_{N-1})$ $(k = 1,\ldots,N-1)$. Let $\gamma = (\gamma_1,\ldots,\gamma_{N-1}) \in \mathbb{R}^{N-1}$, $\boldsymbol{x} = (x_1,\ldots,x_{N-1}) \in \mathbb{R}^{N-1}$. We consider the $GL(N-1,\mathbb{R})$ spectral problem

$$h_k\psi_\gamma(\boldsymbol{x}) = \sigma_k(\gamma)\psi_\gamma(\boldsymbol{x}) \qquad k = 1,\ldots,N-1 \tag{2.5}$$

where $\sigma_k(\gamma)$ are elementary symmetric functions.

Obviously, in the asymptotic region $x_{k+1} \gg x_k$ $(k = 1,\ldots,N-2)$ all potentials vanish and the solution to (2.5) is a superposition of plane waves. The problem is to find a solution to (2.5) satisfying the following properties.

- The solution vanishes very rapidly

$$\psi_\gamma(\boldsymbol{x}) \sim \exp\left\{-\frac{2}{\hbar}\,\mathrm{e}^{(x_k-x_{k+1})/2}\right\} \qquad x_k - x_{k+1} \to \infty. \tag{2.6}$$

- The function ψ_γ is Weyl-invariant, i.e. it is symmetric under any permutation

$$\psi_{\ldots\gamma_j\ldots\gamma_k\ldots} = \psi_{\ldots\gamma_k\ldots\gamma_j\ldots}. \tag{2.7}$$

- ψ_γ can be analytically continued to an entire function of $\gamma \in \mathbb{C}^{N-1}$ and the following asymptotics hold:

$$\psi_\gamma \sim |\gamma_j|^{(2-N)/2}\exp\left\{-\frac{\pi}{2\hbar}(N-2)|\gamma_j|\right\} \tag{2.8}$$

  as $|\operatorname{Re}\gamma_j| \to \infty$ in a finite strip of the complex plane.

Properties (a)–(c) define a unique solution to the spectral problem (2.2).

3. Main results

Theorem 3.1. *The following statements hold [5, 10].*

- *Let a set $\|\gamma_{jk}\|$ be the lower triangular $(N-1)\times(N-1)$ matrix. The solution to the spectral problem (2.5)–(2.8) can be written in the form of multiple Mellin–Barnes integrals*[1]*:*

  $\psi_{\gamma_{N-1,1},\ldots,\gamma_{N-1,N-1}}(x_1,\ldots,x_{N-1})$

[1] We identify the set γ with the last row $(\gamma_{N-1,1},\ldots,\gamma_{N-1,N-1})$.

$$= \frac{(2\pi\hbar)^{-(N-1)(N-2)/2}}{\prod_{k=1}^{N-2} k!} \int_{\mathcal{C}} \prod_{n=1}^{N-2} \frac{\prod_{j=1}^{n} \prod_{k=1}^{n+1} \hbar^{(\gamma_{nj}-\gamma_{n+1,k})/\mathrm{i}\hbar}\, \Gamma\Big(\frac{\gamma_{nj}-\gamma_{n+1,k}}{\mathrm{i}\hbar}\Big)}{\prod_{\substack{j,k=1\\ j<k}}^{n} \Big|\Gamma\Big(\frac{\gamma_{nj}-\gamma_{nk}}{\mathrm{i}\hbar}\Big)\Big|^2}$$

$$\times \exp\left\{\frac{\mathrm{i}}{\hbar} \sum_{n,k=1}^{N-1} x_n \Big(\gamma_{nk} - \gamma_{n-1,k}\Big)\right\} \prod_{\substack{j,k=1\\ j\leqslant k}}^{N-2} \mathrm{d}\gamma_{jk} \tag{3.1}$$

where the integral should be understood as follows: first, we integrate on γ_{11} over the line $\operatorname{Im}\gamma_{11} > \max\{\operatorname{Im}\gamma_{21}, \operatorname{Im}\gamma_{22}\}$; then we integrate on the set $(\gamma_{21}, \gamma_{22})$ over the lines $\operatorname{Im}\gamma_{2j} > \max_m\{\operatorname{Im}\gamma_{3m}\}$ and so on. The last integrations should be performed on the set of variables $(\gamma_{N-2,1} \ldots, \gamma_{N-2,N-2})$ over the lines $\operatorname{Im}\gamma_{N-2,k} > \max_m\{\operatorname{Im}\gamma_{N-1,m}\}$.

- *In the region $x_k \ll x_{k+1} (k = 1, \ldots, N-1)$ the solution has the following asymptotics:*

$$\psi_{\gamma}(\boldsymbol{x}) = \sum_{s\in W} \phi(s\gamma) \mathrm{e}^{\frac{\mathrm{i}}{\hbar}(s\gamma, \boldsymbol{x})} + \mathrm{O}\Big(\max\big\{\mathrm{e}^{x_k - x_{k+1}}\big\}_{k=1}^{N-1}\Big) \tag{3.2}$$

where $(\cdot\,,\cdot)$ is a scalar product in $\mathbb{R}^{N-1}$ and the summation is performed over the permutation group; $\phi(\gamma)$ is a (renormalized) Harish–Chandra function

$$\phi(\gamma) = \hbar^{-2\mathrm{i}(\gamma,\rho)/\hbar} \prod_{j<k} \Gamma\Big(\frac{\gamma_j - \gamma_k}{\mathrm{i}\hbar}\Big) \tag{3.3}$$

where $(\gamma, \boldsymbol{\rho}) \equiv \frac{1}{2}\sum_{m=1}^{N-1}(N-2m)\gamma_k$.

- *The functions (3.1) have the scalar product*

$$\int_{\mathbb{R}^{N-1}} \overline{\psi}_{\gamma'}(\boldsymbol{x})\psi_{\gamma}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \frac{\mu^{-1}(\gamma)}{(N-1)!} \sum_{s\in W} \delta(s\gamma - \gamma') \qquad (\gamma, \gamma' \in \mathbb{R}^{N-1}) \tag{3.4}$$

and obey the completeness condition

$$\int_{\mathbb{R}^{N-1}} \mu(\gamma)\psi_{\gamma}(\boldsymbol{x})\overline{\psi}_{\gamma}(\boldsymbol{y})\,\mathrm{d}\gamma = \delta(\boldsymbol{x} - \boldsymbol{y}) \tag{3.5}$$

where

$$\mu(\gamma) = \frac{(2\pi\hbar)^{1-N}}{(N-1)!} \prod_{j<k} \Big|\Gamma\Big(\frac{\gamma_j - \gamma_k}{\mathrm{i}\hbar}\Big)\Big|^{-2} \tag{3.6}$$

is the Sklyanin measure [3].

The eigenfunctions for the periodic chain are constructed as a kind of Fourier transform with the function (3.1). Let

$$t_N(\lambda; \boldsymbol{E}) = \sum_{k=0}^{N} (-1)^k \lambda^{N-k} E_k \tag{3.7}$$

and $\boldsymbol{e}_j$ denotes the jth basis vector in $\mathbb{R}^{N-1}$.

Theorem 3.2. *The solution to the spectral problem (2.2) and (2.4) can be represented as the integral over real variables $\gamma = (\gamma_1, \ldots, \gamma_{N-1})$ in the following form:*

$$\Psi_{\boldsymbol{E}}(\boldsymbol{x}, x_N) = \frac{1}{2\pi} \int_{\mathbb{R}^{N-1}} \mu(\gamma) C(\gamma; \boldsymbol{E})\, \Psi_{\gamma, E_1}(\boldsymbol{x}, x_N)\,\mathrm{d}\gamma \tag{3.8}$$

where

- *the function* $\Psi_{\gamma,E_1}(\boldsymbol{x}, x_N)$ *is defined in terms of the solution (3.1) to the* $GL(N-1,\mathbb{R})$ *spectral problem:*

$$\Psi_{\gamma,E_1}(\boldsymbol{x}, x_N) = \psi_\gamma(\boldsymbol{x}) \exp\left\{\frac{\mathrm{i}}{\hbar}\left(E_1 - \sum_{m=1}^{N-1}\gamma_m\right)x_N\right\} \tag{3.9}$$

- *the function* $C(\gamma; \boldsymbol{E})$ *is the solution of the multi-dimensional Baxter equations*

$$t_N(\gamma_j; \boldsymbol{E})C(\gamma; \boldsymbol{E}) = \mathrm{i}^N C(\gamma + \mathrm{i}\hbar\boldsymbol{e}_j; \boldsymbol{E}) + \mathrm{i}^{-N} C(\gamma - \mathrm{i}\hbar\boldsymbol{e}_j; \boldsymbol{E}) \tag{3.10}$$

which is a symmetric entire function in γ*-variables with the asymptotics*

$$C(\gamma; \boldsymbol{E}) \sim |\gamma_k|^{-N/2} \exp\left\{-\frac{\pi N|\gamma_k|}{2\hbar}\right\} \tag{3.11}$$

as $\operatorname{Re}\gamma_k \to \pm\infty$ *in the strip* $|\operatorname{Im}\gamma_k| \leqslant \hbar$.

The above restrictions imposed on the solution to (3.10) are a reformulation of the quantization condition (2.4) on the level of a γ-representation. To obtain the explicit integral form for the eigenfunctions, we use the solution to (3.10) and (3.11) in the Pasquier–Gaudin form [15] (see section 7 below)

$$C(\gamma; \boldsymbol{E}) = \prod_{j=1}^{N-1} \frac{c_+(\gamma_j; \boldsymbol{E}) - \xi(\boldsymbol{E})c_-(\gamma_j; \boldsymbol{E})}{\prod_{k=1}^{N} \sinh\frac{\pi}{\hbar}\big(\gamma_j - \delta_k(\boldsymbol{E})\big)} \tag{3.12}$$

where the entire functions $c_\pm(\gamma)$ are two Gutzwiller's solutions [1] of the one-dimensional Baxter equation

$$t(\gamma; \boldsymbol{E})c(\gamma; \boldsymbol{E}) = \mathrm{i}^{-N} c(\gamma + \mathrm{i}\hbar; \boldsymbol{E}) + \mathrm{i}^N c(\gamma - \mathrm{i}\hbar; \boldsymbol{E}) \tag{3.13}$$

and the parameters $\xi(\boldsymbol{E})$, $\boldsymbol{\delta} = (\delta_1(\boldsymbol{E}), \ldots, \delta_N(\boldsymbol{E}))$ satisfy the Gutzwiller conditions (the energy quantization) [1, 15] (see section 7 below). Then the multiple integral (3.8) can be evaluated explicitly. Let $\boldsymbol{y} = (y_1, \ldots, y_N) \in \mathbb{R}^N$ be an arbitrary vector. We denote by $\boldsymbol{y}^{(s)} \equiv (y_1, \ldots, y_{s-1}, y_{s+1}, \ldots, y_N)$ the corresponding vector in $\mathbb{R}^{N-1}$.

Theorem 3.3. *Assuming that* $\delta_j(\boldsymbol{E}) \neq \delta_k(\boldsymbol{E})$*, the solution (3.8) can be written (up to an inessential numerical factor) in the equivalent form*

$$\Psi_{\boldsymbol{E}}(\boldsymbol{x}, x_N) = \sum_{s=1}^{N}(-1)^{N-s} \sum_{\boldsymbol{n}^{(s)}\in\mathbb{Z}^{N-1}} \Delta(\boldsymbol{\delta}^{(s)} + \mathrm{i}\hbar\boldsymbol{n}^{(s)})\, C_+(\boldsymbol{\delta}^{(s)} + \mathrm{i}\hbar\boldsymbol{n}^{(s)})\, \Psi_{\boldsymbol{\delta}^{(s)}+\mathrm{i}\hbar\boldsymbol{n}^{(s)},E_1}(\boldsymbol{x}, x_N) \tag{3.14}$$

where

$$C_+(\gamma) \equiv \prod_{j=1}^{N-1} c_+(\gamma_j; \boldsymbol{E}) \tag{3.15}$$

and $\Delta(\gamma) = \prod_{j>k}(\gamma_j - \gamma_k)$ *is the Vandermonde determinant.*

Remark 3.1. For $N = 2, 3$ and 4, formula (3.14) reproduces the results obtained by Gutzwiller [1].

4. *R*-matrix approach

The Toda chain can be nicely described using the R-matrix approach [3]. It is well known that the Lax operator

$$L_n(\lambda) = \begin{pmatrix} \lambda - p_n & \mathrm{e}^{-x_n} \\ -\mathrm{e}^{x_n} & 0 \end{pmatrix} \tag{4.1}$$

satisfies the commutation relations

$$R(\lambda-\mu)(L_n(\lambda))\otimes I)(I\otimes L_n(\mu)) = (I\otimes L_n(\mu))(L_n(\lambda)\otimes I)R(\lambda-\mu) \tag{4.2}$$

where

$$R(\lambda) = I\otimes I + \frac{\mathrm{i}\hbar}{\lambda}P \tag{4.3}$$

is a rational R-matrix. The monodromy matrix

$$T_N(\lambda) \stackrel{\text{def}}{=} L_N(\lambda)\ldots L_1(\lambda) \equiv \begin{pmatrix} A_N(\lambda) & B_N(\lambda) \\ C_N(\lambda) & D_N(\lambda) \end{pmatrix} \tag{4.4}$$

satisfies the analogous equation

$$R(\lambda-\mu)(T(\lambda)\otimes I)(I\otimes T(\mu)) = (I\otimes T(\mu))(T(\lambda)\otimes I)R(\lambda-\mu). \tag{4.5}$$

In particular, the following commutation relations hold:

$$[A_N(\lambda), A_N(\mu)] = [C_N(\lambda), C_N(\mu)] = 0 \tag{4.6}$$

$$(\lambda-\mu+\mathrm{i}\hbar)A_N(\mu)C_N(\lambda) = (\lambda-\mu)C_N(\lambda)A_N(\mu) + \mathrm{i}\hbar A_N(\lambda)C_N(\mu) \tag{4.7}$$

$$(\lambda-\mu+\mathrm{i}\hbar)D_N(\lambda)C_N(\mu) = (\lambda-\mu)C_N(\mu)D_N(\lambda) + \mathrm{i}\hbar D_N(\mu)C_N(\lambda). \tag{4.8}$$

From (4.5) it can be easily shown that the trace of the monodromy matrix

$$\widehat{t_N}(\lambda) = A_N(\lambda) + D_N(\lambda) \tag{4.9}$$

satisfies the commutation relations $[\widehat{t}(\lambda), \widehat{t}(\mu)] = 0$ and is a generating function for the Hamiltonians of the periodic Toda chain:

$$\widehat{t_N}(\lambda) = \sum_{k=0}^{N}(-1)^k\lambda^{N-k}H_k. \tag{4.10}$$

We reformulate the spectral equations (2.2) as follows:

$$\widehat{t_N}(\lambda)\Psi_{\boldsymbol{E}} = t_N(\lambda; \boldsymbol{E})\Psi_{\boldsymbol{E}} \tag{4.11}$$

where

$$t_N(\lambda; \boldsymbol{E}) = \sum_{k=0}^{N}(-1)^k\lambda^{N-k}E_k. \tag{4.12}$$

On the other hand, it can be easily shown that the operator

$$A_{N-1}(\lambda) \equiv \sum_{k=0}^{N-1}(-1)^k\lambda^{N-k-1}h_k(x_1, p_1; \ldots; x_{N-1}, p_{N-1}) \tag{4.13}$$

is nothing but the generating function for the Hamiltonians h_k of the $GL(N-1)$ Toda chain. Therefore, the $GL(N-1, \mathbb{R})$ spectral equations can be written in the form

$$A_{N-1}(\lambda)\psi_\gamma(\boldsymbol{x}) = \prod_{m=1}^{N-1}(\lambda-\gamma_m)\,\psi_\gamma(\boldsymbol{x}). \tag{4.14}$$

Using the obvious relation

$$C_N(\lambda) = -\mathrm{e}^{x_N} A_{N-1}(\lambda) \tag{4.15}$$

one obtains, as a trivial corollary of (4.14),

$$C_N(\gamma_j)\psi_\gamma(\boldsymbol{x}) = 0 \qquad \forall\, \gamma_j \in \gamma. \tag{4.16}$$

Remark 4.1. Equations (4.16) are an analytical analogue of the notion of 'operator zeros' introduced by Sklyanin [3].

5. Eigenfunctions for the open Toda chain

Suppose that the solution to (4.14) satisfying (2.6)–(2.8) is given. Using the commutation relations (4.7) and (4.8) together with (4.16), it is easy to show that the following relations hold:

$$A_N(\gamma_j)\psi_\gamma = \mathrm{i}^{-N}\,\mathrm{e}^{-x_N}\,\psi_{\gamma-\mathrm{i}\hbar e_j} \tag{5.1a}$$

$$D_N(\gamma_j)\psi_\gamma = \mathrm{i}^{N}\mathrm{e}^{x_N}\,\psi_{\gamma+\mathrm{i}\hbar e_j} \tag{5.1b}$$

($j = 1, \ldots, N-1$) where $\boldsymbol{e}_j$ are jth basis vectors in $\mathbb{R}^{N-1}$. Note that (5.1b) is a corollary of (5.1a) since the quantum determinant of the monodromy matrix (4.4) is unity.

Let us introduce the key object, i.e. the *auxiliary* function

$$\Psi_{\gamma,\epsilon}(x_1, \ldots, x_N) \overset{\text{def}}{=} \psi_\gamma(\boldsymbol{x}) \exp\left\{\frac{\mathrm{i}}{\hbar}\left(\epsilon - \sum_{m=1}^{N-1}\gamma_m\right)x_N\right\} \tag{5.2}$$

where ϵ is an arbitrary parameter. From (4.14), (4.15) and (5.1) it is readily seen that this function satisfies the equations

$$C_N(\lambda)\Psi_{\gamma,\epsilon} = -\mathrm{e}^{x_N}\prod_{j=1}^{N-1}(\lambda - \gamma_j)\ \Psi_{\gamma,\epsilon} \tag{5.3a}$$

$$A_N(\lambda)\Psi_{\gamma,\epsilon} = \left(\lambda - \epsilon + \sum_{m=1}^{N-1}\gamma_m\right)\prod_{j=1}^{N-1}(\lambda - \gamma_j)\ \Psi_{\gamma,\epsilon} + \mathrm{i}^{-N}\sum_{j=1}^{N-1}\Psi_{\gamma-\mathrm{i}\hbar e_j,\epsilon}\prod_{m\neq j}\frac{\lambda - \gamma_m}{\gamma_j - \gamma_m} \tag{5.3b}$$

$$D_N(\lambda)\Psi_{\gamma,\epsilon} = \mathrm{i}^{N}\sum_{j=1}^{N-1}\Psi_{\gamma+\mathrm{i}\hbar e_j,\epsilon}\prod_{m\neq j}\frac{\lambda - \gamma_m}{\gamma_j - \gamma_m}. \tag{5.3c}$$

In particular,

$$\widehat{t_N}(\gamma_j)\Psi_{\gamma,\epsilon} = \mathrm{i}^N\Psi_{\gamma,\epsilon} + \mathrm{i}^{-N}\Psi_{\gamma,\epsilon}. \tag{5.4}$$

The problem is to find the corresponding solution for the $GL(N, \mathbb{R})$ Toda chain using the above information, i.e. in terms of the function $\Psi_{\gamma,\epsilon}(\boldsymbol{x})$ construct the Weyl invariant function $\psi_{\lambda_1,\ldots,\lambda_N}(x_1, \ldots, x_N)$ satisfying the equations

$$A_N(\lambda)\psi_{\lambda_1,\ldots,\lambda_N} = \prod_{k=1}^{N}(\lambda - \lambda_k)\ \psi_{\lambda_1,\ldots,\lambda_N} \tag{5.5a}$$

$$A_{N+1}(\lambda_n)\psi_{\lambda_1,\ldots,\lambda_N} = \mathrm{i}^{-N-1}\,\mathrm{e}^{x_{N+1}}\,\psi_{\lambda_1,\ldots,\lambda_n-\mathrm{i}\hbar,\ldots,\lambda_N} \qquad (n = 1, \ldots, N) \tag{5.5b}$$

and obeying conditions similar to (2.6)–(2.8).

Lemma 5.1 (See [10]). *Let* $\Psi_{\gamma,\epsilon}(\boldsymbol{x}, x_N)$ *be the auxiliary function (5.2). Let* $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_N) \in \mathbb{C}^N$ *be the set of indeterminates. Let*

$$\mu(\gamma) = \frac{(2\pi\hbar)^{1-N}}{(N-1)!} \prod_{j<k} \left| \Gamma\Big(\frac{\gamma_j - \gamma_k}{\mathrm{i}\hbar}\Big) \right|^{-2} \tag{5.6}$$

$$Q(\gamma_1, \ldots, \gamma_{N-1} | \lambda_1, \ldots, \lambda_N) = \prod_{j=1}^{N-1} \prod_{k=1}^{N} h^{(\gamma_j - \lambda_k)/\mathrm{i}\hbar}\, \Gamma\Big(\frac{\gamma_j - \lambda_k}{\mathrm{i}\hbar}\Big). \tag{5.7}$$

Then the Weyl invariant solution to the spectral problem (5.5a) and (5.5b) with the properties discussed above is given by the recurrent formula

$$\psi_{\lambda_1,\ldots,\lambda_N}(x_1, \ldots, x_N) = \int_C \mu(\gamma) Q(\gamma; \boldsymbol{\lambda}) \Psi_{\gamma;\lambda_1+\cdots+\lambda_N}(x_1, \ldots, x_N)\, \mathrm{d}\gamma \tag{5.8}$$

where the integration is performed along the horizontal lines with $\mathrm{Im}\,\gamma_j > \max_k\{\mathrm{Im}\,\lambda_k\}$.

Proof. One needs to calculate the action of the operators $A_N(\lambda)$ and

$$A_{N+1}(\lambda) = (\lambda - p_{N+1}) A_N(\lambda) + \mathrm{e}^{-x_{N+1}} C_N(\lambda) \tag{5.9}$$

on the function (5.8) using the formulae (5.3b) and (5.5a), (5.3a). The shifted contours can be deformed to the original ones by using the fact that the integrand in (5.8) is an entire function which quickly decreases in any finite horizontal strip of the complex plane as $|\mathrm{Re}\,\gamma_j| \to \infty$. The last step is to use the difference equations for the parts of the integrand with respect to the shifts $\pm\mathrm{i}\hbar$ of parameters γ_m and λ_k. □

Proof of theorem 3.1. The proof of (3.1) is a straightforward resolution of the recurrent relations (5.8) starting with the trivial eigenfunction $\psi_{\gamma_{11}}(x_1) = \exp\{\frac{\mathrm{i}}{\hbar}\gamma_{11}x_1\}$. Obviously, this function is symmetric under permutation of the parameters γ. The asymptotics (2.6) can be proved using the steepest-descent method. Using the Stirling formula for the Γ-functions as $\gamma_{N-1,k} \equiv \gamma_k \to \pm\infty$, it is easy to see that the asymptotics (2.8) hold. Hence, equation (3.1) is an appropriate solution to the spectral problem.

Furthermore, formula (3.2) can be proved as follows. The integrand in (5.8) decreases exponentially as $\gamma_j \to -\mathrm{i}\infty$ ($j = 1, \ldots, N-1$) and, consequently, the integrals over large semi-circles in the lower half-plane vanish. Using the Cauchy formula to calculate the integral (5.8) in the asymptotic region $x_{k+1} \gg x_k$ ($k = 1, \ldots, N-1$), it is easy to see that the asymptotics of the function ψ_γ are determined precisely in terms of the corresponding Harish–Chandra function (3.3).

The scalar product (3.4) is a consequence of the Plancherel formula proved in [16] for the $SL(N, \mathbb{R})$ case. Formula (3.5) can be proved by induction. □

Remark 5.1. In [5] (equations (4.18) and (4.7)) the eigenfunction (3.1) was constructed in terms of a Weyl-invariant Whittaker function (coincidence can be shown by comparing asymptotics and analytical properties of both functions). The Whittaker function possesses a standard integral representation corresponding to the Iwasawa decomposition of a semisimple group (see, for example, [9]). It differs from our one. So one can consider the representation (3.1) as a new one for the Whittaker function.

6. Periodic chain: γ-representation, eigenfunctions and the Plancherel formula

Let $\Psi_{\boldsymbol{E}}(\boldsymbol{x}, x_N)$ be the quickly decreasing solution of the problem (4.11). We define the function $C(\gamma; \boldsymbol{E})$ by the generalized Fourier transform

$$\delta(E_1 - \epsilon)\, C(\gamma; \boldsymbol{E}) = \int_{\mathbb{R}^{N-1}} \Psi_{\boldsymbol{E}}(\boldsymbol{x}, x_N)\, \overline{\Psi}_{\gamma,\epsilon}(\boldsymbol{x}, x_N)\, \mathrm{d}\boldsymbol{x}\, \mathrm{d}x_N. \tag{6.1}$$

Lemma 6.1. *The function $C(\gamma)$ possesses the following properties:*

(a) it is a symmetric function with respect to the γ-variables;
(b) it is an entire function of $\gamma \in \mathbb{C}^{N-1}$;
(c) the function $C(\gamma)$ obeys the asymptotics

$$C(\gamma; \boldsymbol{E}) \sim |\gamma_k|^{-N/2} \exp\left\{-\frac{\pi N|\gamma_k|}{2\hbar}\right\} \tag{6.2}$$

as $\operatorname{Re}\gamma_k \to \pm\infty$ *in the strip* $|\operatorname{Im}\gamma_k| \leqslant \hbar$*;*
(d) the function $C(\gamma)$ satisfies the multi-dimensional Baxter equation

$$t(\gamma_j; \boldsymbol{E})C(\gamma; \boldsymbol{E}) = \mathrm{i}^N C(\gamma + \mathrm{i}\hbar \boldsymbol{e}_j; \boldsymbol{E}) + \mathrm{i}^{-N} C(\gamma - \mathrm{i}\hbar \boldsymbol{e}_j; \boldsymbol{E}) \tag{6.3}$$

where $t(\gamma; \boldsymbol{E})$ is defined by (4.12).

Proof. The symmetry of the function $C(\gamma)$ is obvious. We present here only a sketch of the proof of statements (b) and (c). Statement (b) follows from the assertion that the auxiliary function $\Psi_{\gamma,\epsilon}$ is an entire one, while the solution of the periodic chain vanishes very rapidly as $|x_k - x_{k+1}| \to \infty$ [2]. (c) The asymptotics (6.2) are a combination of two factors. The first one comes from the asymptotics (2.8), while the additional factor $\sim |\gamma_k|^{-1} \exp\{-\pi|\gamma_k|/\hbar\}$ results from the stationary phase method while calculating the multiple integral including the function (3.1). The calculation is based heavily upon the exact asymptotics of the function $\Psi_{\boldsymbol{E}}(\boldsymbol{x}, x_N)$ as $|x_k - x_{k+1}| \to \infty$.

The proof of (d) is simple. Using the definition (4.11) and integrating by parts (evidently, boundary terms vanish), one obtains

$$\begin{aligned}\delta(E_1 - \epsilon)\, t(\gamma_j; \boldsymbol{E})C(\gamma) &\equiv \int_{\mathbb{R}^{N-1}} \left\{\widehat{t}(\gamma_j)\Psi_{\boldsymbol{E}}(\boldsymbol{x}, x_N)\right\} \overline{\Psi}_{\gamma,\epsilon}(\boldsymbol{x}, x_N)\, \mathrm{d}\boldsymbol{x}\, \mathrm{d}x_N \\ &= \int_{\mathbb{R}^{N-1}} \Psi_{\boldsymbol{E}}(\boldsymbol{x}, x_N)\, \overline{\widehat{t}(\gamma_j)\Psi_{\gamma,\epsilon}(\boldsymbol{x}, x_N)}\, \mathrm{d}\boldsymbol{x}\, \mathrm{d}x_N.\end{aligned} \tag{6.4}$$

Taking into account the relation (5.4), the Baxter equation (6.3) follows from definition (6.1). □

Proof. Now we prove theorem 3.2. Using the completeness condition

$$\int_{\mathbb{R}^N} \mu(\gamma)\Psi_{\gamma,\epsilon}(\boldsymbol{x}, x_N)\overline{\Psi}_{\gamma,\epsilon}(\boldsymbol{y}, y_N)\, \mathrm{d}\gamma\, \mathrm{d}\epsilon = 2\pi\hbar\delta(\boldsymbol{x} - \boldsymbol{y})\delta(x_N - y_N) \tag{6.5}$$

which is a corollary of (3.5), the inversion of formula (6.1) results in the expression

$$\Psi_{\boldsymbol{E}}(\boldsymbol{x}, x_N) = \frac{1}{2\pi} \int_{\mathbb{R}^{N-1}} \mu(\gamma)C(\gamma; \boldsymbol{E})\, \Psi_{\gamma,E_1}(\boldsymbol{x}, x_N)\, \mathrm{d}\gamma. \tag{6.6}$$

[2] Actually, the boundary conditions have the same importance here as the requirement of compact support in the theory of analytic continuation for the usual Fourier transform.

The integral (6.6) is correctly defined. Indeed, the measure (3.6) is an entire function. Therefore, there are no poles in the integrand. Moreover,

$$\mu(\gamma) \sim |\gamma_k|^{N-2} \exp\left\{\frac{\pi}{\hbar}(N-2)|\gamma_k|\right\} \tag{6.7}$$

as $|\gamma_k| \to \infty$. Taking into account the asymptotics (2.8) and (6.2) one concludes that the integrand has the behaviour $\sim |\gamma_k|^{-1} \exp\{-\pi|\gamma_k|/\hbar\}$ as $|\gamma_k| \to \infty$. Therefore, the integral (6.5) is convergent. One can directly prove the spectral problem (4.11) by calculating the action of the operator $\widehat{t_N}(\lambda) = A_N(\lambda) + D_N(\lambda)$ on the right-hand side of (6.6) with the help of the formulae (5.3*b*) and (5.3*c*). The calculation is performed similarly to those of lemma 5.1, using the analytical properties of the integrand and the Baxter equation (6.3) (see [5] for details).

The last step is to prove that the function (6.6) satisfies the integrability requirement (2.4). Using the scalar product

$$\int_{\mathbb{R}^N} \overline{\Psi}_{\gamma',\epsilon'}(\boldsymbol{x}, x_N)\Psi_{\gamma,\epsilon}(\boldsymbol{x}, x_N)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}x_N = (2\pi\hbar)\frac{\mu(\gamma)}{(N-1)!}\,\delta(\epsilon-\epsilon')\sum_{s\in W}\delta(s\gamma-\gamma') \tag{6.8}$$

one can write the Plancherel formula

$$2\pi\hbar\int_{\mathbb{R}^N} \overline{\Psi}_{\boldsymbol{E}'}(\boldsymbol{x}, x_N)\Psi_{\boldsymbol{E}}(\boldsymbol{x}, x_N)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}x_N = \delta(E_1-E_1')\int_{\mathbb{R}^{N-1}} \mu(\gamma)\,\overline{C}(\gamma;\boldsymbol{E}')C(\gamma;\boldsymbol{E})\,\mathrm{d}\gamma. \tag{6.9}$$

The integral on the right-hand side of (6.9) is absolutely convergent due to asymptotics (6.2) and (6.7). Hence, the norm $\|\Psi_{\boldsymbol{E}}\|$ is finite modulo the $GL(1)$ δ-function $\delta(E_1-E_1')$ (see the corresponding factor in (2.3) which leads to this function) and the requirement (2.4) is fulfilled. Hence, theorem 3.2 is proved. □

7. Solution of the Baxter equation

It is well known [1, 15] (see also [5] for details) that the solution to the Baxter equation (3.10) with the asymptotics (3.11) can be written in the following separated form:

$$C(\gamma;\boldsymbol{E}) = \prod_{j=1}^{N-1}\frac{c_+(\gamma_j;\boldsymbol{E}) - \xi c_-(\gamma_j;\boldsymbol{E})}{\prod_{k=1}^{N}\sinh\frac{\pi}{\hbar}(\gamma_j-\delta_k)} \tag{7.1}$$

where ξ and δ_k are arbitrary constants and the entire functions $c_\pm(\gamma)$ are defined in terms of $\mathbb{N}\times\mathbb{N}$ determinants:

$$c_+(\gamma) = \frac{1}{\prod_{k=1}^{N}\hbar^{-\mathrm{i}\gamma/\hbar}\Gamma(1-\frac{\mathrm{i}}{\hbar}(\gamma-\lambda_k))} \times \begin{vmatrix} 1 & \frac{1}{t(\gamma+\mathrm{i}\hbar)} & 0 & \dots & \dots\dots \\ \frac{1}{t(\gamma+2\mathrm{i}\hbar)} & 1 & \frac{1}{t(\gamma+2\mathrm{i}\hbar)} & 0 & \dots\dots \\ 0 & \frac{1}{t(\gamma+3\mathrm{i}\hbar)} & 1 & \frac{1}{t(\gamma+3\mathrm{i}\hbar)} & 0\ \dots \\ \dots & \dots & \dots & \dots & \dots\dots \end{vmatrix} \tag{7.2a}$$

$$c_-(\gamma) = \frac{1}{\prod_{k=1}^{N} \hbar^{\mathrm{i}\gamma/\hbar}\Gamma(1+\frac{\mathrm{i}}{\hbar}(\gamma-\lambda_k))}$$
$$\times \begin{vmatrix} \cdots\cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots\ 0 & \frac{1}{t(\gamma-3\mathrm{i}\hbar)} & 1 & \frac{1}{t(\gamma-3\mathrm{i}\hbar)} & 0 \\ \cdots\cdots & 0 & \frac{1}{t(\gamma-2\mathrm{i}\hbar)} & 1 & \frac{1}{t(\gamma-2\mathrm{i}\hbar)} \\ \cdots\cdots & \cdots & 0 & \frac{1}{t(\gamma-\mathrm{i}\hbar)} & 1 \end{vmatrix} \tag{7.2b}$$

and $\lambda_k \equiv \lambda_k(\boldsymbol{E})$ are the roots of the polynomial $t(\gamma) \equiv t_N(\gamma; \boldsymbol{E})$.

On the other hand, the solution (7.1) is not an entire function in general since the denominator in (7.1) has an infinite number of poles at $\gamma = \delta_k + \mathrm{i}\hbar n_k$, $n_k \in \mathbb{Z}$, $k = 1, \ldots, N$. The poles are cancelled only if the following conditions hold:

$$c_+(\delta_k + \mathrm{i}\hbar n_k) = \xi c_-(\delta_k + \mathrm{i}\hbar n_k). \tag{7.3}$$

In turn, this means that the Wronskian

$$W(\gamma) = c_+(\gamma)c_-(\gamma+\mathrm{i}\hbar) - c_+(\gamma+\mathrm{i}\hbar)c_-(\gamma) \tag{7.4}$$

vanishes at $\gamma = \delta_k + \mathrm{i}\hbar n_k$. The Wronskian is an $\mathrm{i}\hbar$ periodic function and obtains exactly N real roots $\delta_k(\boldsymbol{E})$ [1]. Therefore, the solution (7.1) has no poles if one takes $\delta_k = \delta_k(\boldsymbol{E})$ provided that the constant ξ is chosen in such a way that

$$\xi = \left.\frac{c_+(\gamma)}{c_-(\gamma)}\right|_{\gamma=\delta_k(\boldsymbol{E})} \qquad k = 1, \ldots, N. \tag{7.5}$$

Hence, one arrives at the following:

Lemma 7.1 (See [15]). *The function*

$$C(\gamma; \boldsymbol{E}) = \prod_{j=1}^{N-1} \frac{c_+(\gamma_j; \boldsymbol{E}) - \xi(\boldsymbol{E})c_-(\gamma_j; \boldsymbol{E})}{\prod_{k=1}^{N} \sinh\frac{\pi}{\hbar}\big(\gamma_j - \delta_k(\boldsymbol{E})\big)} \tag{7.6}$$

where $\delta_k(\boldsymbol{E})$ are real zeros of the Wronskian (7.4) and the constant ξ is chosen according to (7.5), satisfies the conditions of lemma 6.1.

The quantization conditions

$$\frac{c_+(\delta_1)}{c_-(\delta_1)} = \cdots = \frac{c_+(\delta_N)}{c_-(\delta_N)} \tag{7.7}$$

determine the energy spectrum of the problem. They have been obtained for the first time by Gutzwiller [1] using quite a different method.

Proof. To prove theorem 3.3, one should substitute the solution (7.6) into the integral formula (3.8) and calculate the residues coming from individual terms

$$\frac{c_\pm(\gamma_j; \boldsymbol{E})}{\prod_{k=1}^{N} \sinh\frac{\pi}{\hbar}\big(\gamma_j - \delta_k(\boldsymbol{E})\big)}. \tag{7.8}$$

The result is exactly the sum over all possible poles of expressions (7.8) and essentially coincides with (3.14) (see the careful analysis in [5]). □

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